Int J Theor Phys (2008) 47: 1312–1315
DOI 10.1007/s10773-007-9565-1

Riemann-Christoffel Flows

Patricio S. Letelier

Received: 1 August 2007 / Accepted: 11 September 2007 / Published online: 3 October 2007

Abstract A geometric flow based in the Riemann-Christoffel curvature tensor that in two dimensions has some common features with the usual Ricci flow is presented. For n dimensional spaces this new flow takes into account all the components of the intrinsic curvature. For four dimensional Lorentzian manifolds it is found that the solutions of the Einstein equations associated to a "detonant" sphere of matter, as well, as a Friedman-Roberson-Walker cosmological model are examples of Riemann-Christoffel flows. Possible generalizations are mentioned.

Keywords Curvature tensor · Einstein equations · Detonant matter · Cosmological models

The Ricci flow

$$\frac{\partial g_{ab}}{\partial \lambda} = R_{ab}, \tag{1}$$

was introduced by Hamilton [1] in 1982 and almost immediately applied to the classification of 3-manifolds. This work has its zenith with the celebrated results of Perelman [2] that has a notable recognition. A nice introduction to Ricci flows can be found in [3].

Motivated by different mathematical and physical applications the original Ricci flow formula suffered many generalizations that can be classified in two groups. In the first, new terms are added to the Ricci tensor, e.g., $(kR + \Lambda)g_{ab}$ where k and Λ are constants and R is the Ricci scalar, and/or a symmetric tensor built with vector or scalar fields like $\xi_a\xi_b$, $\nabla_a\Phi\nabla_b\Phi$, and $\nabla_a\nabla_b\Phi$ [4–6]. In the second, besides $\frac{\partial g_{ab}}{\partial \lambda}$, terms containing second derivatives with respect to λ are considered in order to have a hyperbolic or elliptic partial differential equation rather than a parabolic one [7, 8].

The mathematical applications are mainly concentrated in the study of Ricci flows for metrics of two or three dimensional spaces. The physical applications deal more with four or higher dimensional spaces.

P.S. Letelier (✉)
Departamento de Matemática Aplicada-IMECC, Universidade Estadual de Campinas,
13081-970 Campinas, S.P., Brazil
e-mail: letelier@ime.unicamp.br

For two and three dimensional manifolds the Ricci tensor characterizes completely the intrinsic curvature of these spaces. Only in these dimensions, once the Ricci tensor is given one can find all the components of the Riemann-Chistoffel tensor. Moreover, loosely speaking, we have that for large n the number of components of the Ricci tensor grows like n^2 whereas the number of components of the Riemann-Chistoffel tensor increases like n^4.

Thus, when one applies the usual Ricci flow formula, or any of the generalizations above mentioned, for spaces of four or more dimensions we have that the Ricci tensor only carries partial information about the space intrinsic curvature.

Motivated by these considerations I shall introduce a new geometric flow based in the Riemann-Christoffel curvature tensor instead of the Ricci tensor. Let me define the Riemann-Christoffel "flow" by the expression,

$$\frac{\partial g_{abcd}}{\partial \lambda} = R_{abcd}, \tag{2}$$

$$g_{abcd} \equiv g_{ac} g_{bd} - g_{ad} g_{bc}. \tag{3}$$

The convention used are: for the Riemann-Christoffel tensor $R^a_{bcd} = \Gamma^a_{bc,d} - \cdots$ and for the Ricci tensor $R_{bd} = R^a_{bad}$. Really, (2) does not have the usual form of a usual flow $[\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})]$, but it can be cast as,

$$\frac{\partial g_{ij}}{\partial \lambda} H^{ij}{}_{abcd} = R_{abcd}, \tag{4}$$

with

$$H^{ij}{}_{abcd} = \delta^{(i}_a \delta^{j)}_c g_{bd} + \delta^{(i}_b \delta^{j)}_d g_{ac} - \delta^{(i}_a \delta^{j)}_d g_{bc} - \delta^{(i}_b \delta^{j)}_c g_{ad}, \tag{5}$$

where $\delta^{(i}_a \delta^{j)}_b = \frac{1}{2}(\delta^i_a \delta^j_b + \delta^j_b \delta^i_b)$. A system of equations like (4) may be named a generalized flow.

To compare this Riemann-Christoffel generalized flow with the usual Ricci flow let me first consider the identity:

$$H^{ij}{}_{ab} \equiv H^{ijc}{}_{acb} \tag{6}$$

$$= (n-2)\delta^{(i}_a \delta^{j)}_b + g_{ab} g^{ij}. \tag{7}$$

Now, from (7) and (4) we find,

$$(n-2)\frac{\partial g_{ab}}{\partial \lambda} + g_{ab} g^{ij} \frac{\partial g_{ij}}{\partial \lambda} = R_{ab}. \tag{8}$$

Since a two dimensional space is always an Einstein space we have that for $n = 2$, (8) reduces to the single scalar equation,

$$\frac{1}{g}\frac{\partial g}{\partial \lambda} = \frac{1}{2} R, \tag{9}$$

where g denotes the determinant of the metric, g_{ij}. An equivalent equation can be obtained from (1) modulo the factor one half that can be absorbed redefining the parameter λ. We note that (2) in two dimensions gives us only one independent equation that is equivalent to (9) contrary to the Ricci flow that gives us three. To be more specific for the metric,

$$ds^2 = e^{a(x,y,\lambda)}(dx^2 + dy^2), \tag{10}$$

the Riemann-Christoffel flow gives us the single equation,

$$4a_{,\lambda}e^{a} = a_{,xx} + a_{,yy}, \tag{11}$$

that, modulo a redefinition of λ, is the same well known equation obtained from the Ricci flow. In this case, we have for the Ricci flow that one of the equations is identically null and the other two are equal. For the other frequently used form of the two dimensional metric,

$$ds^2 = dx^2 + [A(x, y, \lambda)]^2 dy^2, \tag{12}$$

the Riemann-Christoffel flow gives,

$$2A_{,\lambda} = A_{,xx}. \tag{13}$$

For the Ricci flow we have only two different equations, the first is exactly the same equation (13) and the second is $A_{,xx} = 0$, that kills the flow. Note that, in this case, the Riemann-Christoffel flow obeys exactly a linear heat equation.

Now let us comeback to the general case, $n \geq 3$, from (8) and the tensor,

$$K^{ab}{}_{kl} = \frac{1}{n-2}\left[\delta_k^{(a}\delta_l^{b)} - \frac{1}{2(n-1)}g^{ab}g_{kl}\right], \tag{14}$$

that has the property

$$H^{ij}{}_{ab}K^{ab}{}_{kl} = \delta_k^{(i}\delta_l^{j)}, \tag{15}$$

we find

$$(n-2)\frac{\partial g_{ab}}{\partial \lambda} = R_{ab} - \frac{1}{2(n-1)}g_{ab}R. \tag{16}$$

For $n = 3$ this last equation is equivalent to the Riemann-Christoffel flow, we have that only for spaces with these dimensions the Ricci tensor has the same number of components than the Riemann-Christoffel curvature tensor, six.

For spaces with constant curvature, K, in any dimensions we have,

$$R_{abcd} = -K g_{abcd}. \tag{17}$$

Thus the solution of (2) is

$$g_{abcd}(x, \lambda) = e^{-K\lambda} g_{abcd}(x, 0). \tag{18}$$

Hence,

$$g_{ab}(x, \lambda) = e^{-\frac{1}{2}K\lambda} g_{ab}(x, 0). \tag{19}$$

We have contraction for spaces of positive curvature and expansion for negative curvature.

Now I shall study examples of particular Riemann-Christoffel geometric flows for four dimensions Lorentzian manifolds.

Let me consider the particular spherically symmetric metric,

$$ds^2 = [h(t, r, \lambda)]^2 dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{20}$$

Equation (2), in this case, reduces to

$$2h_{,\lambda} = -h_{,rr}, \qquad 2h_{,\lambda} = -h_{,r}/r, \tag{21}$$

that has the general solution $h = f(t)(\lambda - r^2)$, where f is an arbitrary function of the indicated argument. If one regards this spacetime as a solution of the Einstein field equations,

$$R_{ab} - \frac{1}{2} g_{ab} R = -T_{ab}, \tag{22}$$

we find a diagonal energy-momentum tensor with $T_0^0 = 0$ and $T_1^1 = T_2^2 = T_3^3 = p$, with $p = 4/(\lambda - r^2)$. Therefore we have a perfect "fluid" with pressure for $\lambda > r^2$ or tension when $\lambda < r^2$, and no energy density. This fluid may be called a "detonant fluid", since in a detonation the pressure is by far more important than the density and this last may be disregarded. We note a dilution of the pressure as λ increases. We have a sort of phase transition in the shell of radius $r_c = \sqrt{\lambda}$. In this case the parameter λ may be associated to a thermodynamic variable.

Now, I shall consider Riemann-Christoffel geometrical flows for Friedman-Roberson-Walker cosmological models with flat spacial sections,

$$ds^2 = a^{-1}(t, \lambda) dt^2 - a(t, \lambda)(dx^2 + dy^2 + dz^2). \tag{23}$$

Equations (2) give us,

$$8a_{,\lambda} = a_{,t}^2, \quad a_{,tt} = 0, \tag{24}$$

whose solution is $a = kt + k^2\lambda/8$, where k is an arbitrary constant. From the Einstein equations we find that the source of this space time is a perfect fluid with density $\rho = 6k/(k\lambda + 8t)$ and pressure $p = \rho/3$, i.e., we have an expanding universe filled by a photon gas. We have a dilution of the pressure and density when λ increases.

In summary, the Riemann-Christoffel geometric flow (2), in two dimensions has some common features with the usual Ricci flow (1). For n dimensional spaces this flows take into account all the component of the intrinsic curvature. For four dimensional Lorentzian manifolds I find that the solutions of the Einstein equations associated to a very simple "detonant sphere" of matter, as well as, a flat Friedman-Robertson-Walker cosmological with $p = \rho/3$ equation of state are examples of Riemann-Christoffel geometric flows. Incidentally, these two examples do not satisfy the Ricci flow (1). For manifolds of dimension $n \geq 4$ a similar flow can be constructed with the Weyl tensor, W_{abcd}, that has no meaning for $n < 4$. For $n = 4$ the Levi-Civita totally anti-symmetric tensor, ε_{abcd}, can be used to add new terms to the Riemann-Christoffel flow. These new terms may involve new fields or other geometrical quantities e.g., $\varepsilon_{abcd}(kR + \Lambda)$. Also extension of the Riemann-Christoffel geometric flow involving second derivatives of the parameter λ may be considered.

Acknowledgements I want to thank R. Mosna, C. Negreiros, S.R. Oliveira, and A. Saa for discussions about Ricci flows and also FAPESP and CNPq for partial financial support.

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